Why and how geometric algebra should be taught at high school : Experiences and proposals (Innovative Teaching of Mathematics with Geometric Algebra)

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Why and how geometric algebra should be taught at high school. Experiences and proposals.

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Research as a source of new pedagogic methods in mathematics

There is a close dependence between pedagogic methods and lines of research. In fact, there exists a continuous interchange between new advances in science and new methods of teaching it. The present lines of research will become the new subjects to teach. I may give some examples: differential calculus, algebra, arithmetic, etc.

First example: differential calculus

Newton and Leibniz discovered differential (and integral) calculus, which was also developed by other researchers such as L'Hôpital, Bernoulli and Euler. In the XVII and XVIII centuries, differential calculus was an active field of research. In the XIX century it becomes an academic subject at university level. In the XX century Newton's fluxions (derivatives) and Leibniz's integrals and differentials are taught to 16-17-year-old pupils at high schools. Aside of the main relation between the derivative and the primitive, I enumerate other theorems taught at these levels:

Chain rule for derivatives of composite functions.
Barrow's rule for the definite integral.
Rolle's theorem.
Mean value theorem.
Cauchy's theorem.
L'Hôpital's rule to calculate indeterminate limits.

Second example: linear algebra

In the field of linear algebra, let us recall that matrices were discovered by Arthur Cayley[1] although Leibniz already worked with determinants[2]. Nowadays, foundations of linear algebra are taught at high schools:

Operations with matrices
Determinants.
Solving the systems of linear equations with determinants.
Gauss' method to solve systems and invert matrices.
Rouché-Frobenius theorem.

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[1] *A memoir on the theory of matrices* (1858).
[2] See E. J. Aiton, *Leibniz, a biography*, Adam Hilger Ltd. (Redcliffe Way, 1985), pp. 125-127. Leibniz outlined a definitive formulation of the rules of determinants in a manuscript dated in January 22nd, 1684.

Third example: Ancients' geometry

We also have other examples. Archimedes discovered that the volumes of the cone, sphere and cylinder having the same diameter and altitude are in the ratio 1:2:3. This aphorism was engraved in his tomb. Geometry was an active field of research during the Hellenistic period. Now we are teaching the formulas of the volumes to 12-14-year-old pupils but also other geometric theorems such as:

Theorems about angles.
Theorem of the isosceles triangle.
Formulas to calculate the area of planar figures.
Formulas to calculate volumes.

Fourth example: arithmetic

The field of arithmetic has also been incorporated to the curricula of high schools. Among others we teach (12-year-old pupils):

Fundamental theorem of arithmetic (decomposition of a number in prime factors).
Maximum common divisor and minimum common multiple.
Divisibility criteria.
Fractions[3].
Decimal fractions[4].

In algebra, nowadays we use the algebraic notation mainly developed by Viète.

As a last example, I teach in the subject of computer science the binary numbers, which were firstly worked by Leibniz[5].

So we must expect that any new field of mathematics, now object of research, in a little time will become an academic subject, also at high school. We may remember Ernst Haeckel's aphorism: "ontogeny recapitulates phylogeny". Although this statement is nowadays considered a falsehood, we may adapt its sense and say:

"<u>Teaching mathematics recapitulates its history</u>"

The field of geometric algebra will also follow this path. Those new areas now under the scope of research will be soon learned by our pupils also at high school. In fact the main problem of geometric algebra is the blockade it suffers mainly at university (at least in Spain, although we have the feeling that this is a general situation all over the world). So our contributions to geometric algebra will soon become an academic subject if we are able to overcome this blockade.

However every coin has two sides and this is also applicable to the interaction between pedagogy and research. My experience teaching mathematics (16 years) has

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[3] Already used by the Egyptians. See B. L. van der Waerden, *Geometry and Algebra in Ancient Civilizations*, p. 165. Springer Verlag (Berlin, Heidelberg, 1983).

[4] Simon Stevin introduced the decimal positional fractions in *De Thiende* (*The Dime*) in 1585.

[5] G. W. Leibniz, « Explication de l'arithmetique binaire, qui se sert des seuls caractères 0 et 1, avec des remarques sur son utilité, et sur ce qu'elle donne le sens des anciennes figures chinoises de fohy », Memoir de l'Acad. des Sciences (1703), *Mathematische Schriften*, vol. II, p. 223, Georg Olms Verlag (Hildesheim, 1971).

shown me that the process of teaching is a constant source of new ideas and inspiration for those that are working in research. My lessons on geometry, analysis and linear algebra at high school have supplied me with many new ideas to advance in the research of geometric algebra. I may affirm that the *Treatise of plane geometry through geometric algebra* would not exist if I had not been a teacher of mathematics because of the lack of the source of inspiration.

The title of the conference is:

"How and why geometric algebra should be taught at high school. Experiences and proposals."

because we must accelerate the collapse of the barriers preventing the geometric algebra from being reckoned an important academic subject, the high school being a first and main front. At this level, the mind of our pupils is very flexible and receptive. Any seed of geometric algebra sown at this age will give very successful fruits whenever there exists continuity of the teaching at university level. The main question is not the difficulty of geometric algebra for our pupils, since any topic may be adapted to the corresponding level. The true problem is the inertia of the departments (of the teachers themselves), which tend to impart the same subjects that have been always taught. However do not think that these subjects are a good sample of all the mathematics. Not at all, but usually they are a partial and slanted view of them. So the renovation, the up-to-date reintroduction of geometry through geometric algebra will be a hard task and likely we shall have to battle against our own colleagues.

Now I give some examples about how we may teach parts of geometric algebra to our students. I am mainly considering 16-17-year-old pupils, but also you may lengthen this range to the first courses of university. The transition from high school to college should be more gradual than the current one. In our school we intend that pupils do not suffer a crack in mathematics during this transition and I believe that in some degree we are achieving this purpose. However, if we introduce concepts of geometric algebra at high school and they do not have continuity at university, what is the yield of this effort?

Barycentric coordinates

One of the most interesting approaches to point geometry is the use of barycentric coordinates. Every point on a line may be written as linear combination of two given points of this line (figure 1):

$$R=(1-k)P+kQ$$

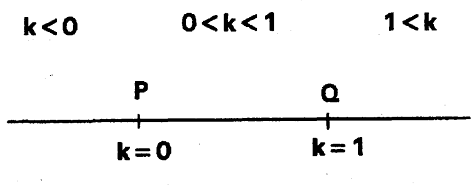


Figure 1

with coefficients whose addition is the unity. These coefficients are the barycentric coordinates. In the same way, the linear combination of two lines is the pencil of lines passing through the intersection of both lines. Every line of the pencil (figure 2) has the following general equation:

$$(1-p)(n_1x+n_2y+c)+p\,(n'_1x+n'_2y+c')=0$$

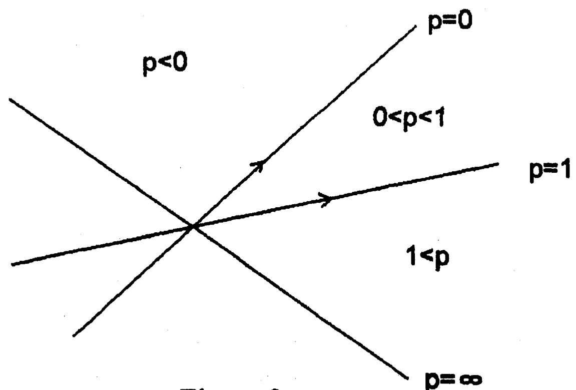


Figure 2

In the space, we may write any plane belonging to an axial pencil as linear combination of two planes of this pencil. Consider the problem of finding the plane that contains a line and an outer point (figure 3). If the line is given as intersection of two planes, the best way to solve this problem is to use the axial pencil of this line. Example:

Find the plane containing the line

$$\begin{cases} 2x+3y=4 \\ x+y-z=2 \end{cases} \text{ and the point } (3, 4, -2).$$

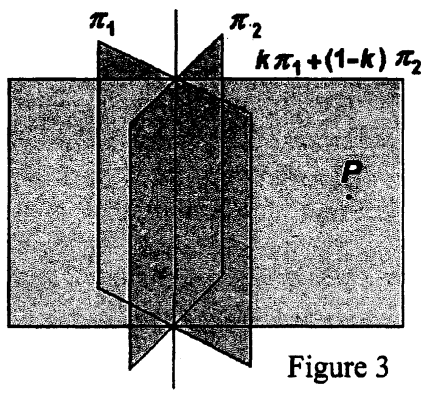


Figure 3

Take the axial pencil for this line. Every plane of this pencil has the equation (using barycentric coordinates):

$$k\,(2x+3y-4)+(1-k)(x+y-z-2)=0$$

but it must contain the point:

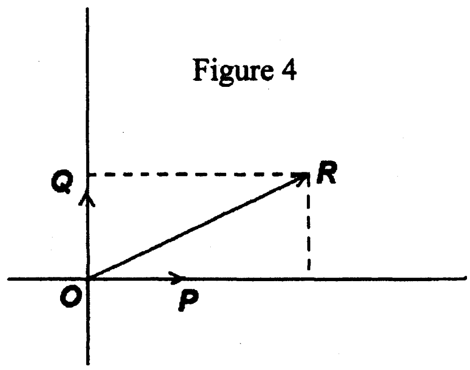
$$k\,(2\cdot 3+3\cdot 4-4)+(1-k)(3+4+2-2)=0 \qquad \Rightarrow \qquad k=-1$$

and so the searched plane is: $y+2z=0$

In fact, a system of Cartesian coordinates hides the barycentric coordinates of points (figure 4):

$$R=(x, y)=O+x\,OP+y\,OQ=(1-x-y)O+x\,P+y\,Q$$

Figure 4



That is, every point R can be written as linear combination of three non-aligned points $\{O, P, Q\}$ with barycentric coordinates always summing the unity.

An immediate application of barycentric coordinates is the calculus of the oriented area S of a triangle ABC:

$$S_{ABC}=\frac{1}{2}AB\wedge BC=\frac{1}{2}\begin{vmatrix} 1-x_A-y_A & x_A & y_A \\ 1-x_B-y_B & x_B & y_B \\ 1-x_C-y_C & x_C & y_C \end{vmatrix} e_{12}$$

which vanishes if the three points are aligned.

In the same way, the oriented volume of the tetrahedron having vertexes A, B, C and D may be written using barycentric coordinates:

$$V=\frac{1}{6}AB\wedge BC\wedge CD=\frac{1}{6}\begin{vmatrix}1-x_A-y_A-z_A & x_A & y_A & z_A\\ 1-x_B-y_B-z_B & x_B & y_B & z_B\\ 1-x_C-y_C-z_C & x_C & y_C & z_C\\ 1-x_D-y_D-z_D & x_D & y_D & z_D\end{vmatrix}e_{123}$$

These coordinates are very useful in projective geometry. A projectivity is simply defined as a linear transformation of the barycentric coordinates[6]:

$$\begin{pmatrix}1-x'-y'\\ x'\\ y'\end{pmatrix}=k\begin{pmatrix}h_{00} & h_{01} & h_{02}\\ h_{10} & h_{11} & h_{12}\\ h_{20} & h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}1-x-y\\ x\\ y\end{pmatrix} \qquad \det h_{ij}\neq 0$$

where k is a variable number allowing that the transformed barycentric coordinates have a sum equal to the unity.

Dual coordinates in the dual plane

The concept of pencil of lines leads to the definition of the dual plane, whose barycentric coordinates are those of the pencils of lines. Given a point base $\{O,P,Q\}$ the dual base is formed by the lines $\overline{OP}$, $\overline{PQ}$ and $\overline{QO}$. For Cartesian coordinates, the dual base is $\{-x-y+1=0,\ x=0,\ y=0\}$. Every line on the plane is expressed as a linear combination of these three lines by means of barycentric coordinates. Let us calculate the dual Cartesian coordinates of the line $2x + 3y + 4 = 0$. We must solve the identity:

$$2x+3y+4\equiv a'(-x-y+1)+b'x+c'y \qquad \forall x,y$$

$$x(2+a'-b')+y(3+a'-c')+4-a'\equiv 0$$

whose solution is:

$$a'=4 \qquad b'=6 \qquad c'=7$$

Dividing by the sum of coefficients we obtain:

$$\frac{2x+3y+4}{17}\equiv\frac{4}{17}(-x-y+1)+\frac{6}{17}x+\frac{7}{17}y$$

whence the dual coordinates of this line are obtained as $[b, c] = [6/17, 7/17]$. Let us see their meaning. The linear combination of both coordinate axes is a line of the pencil of lines passing through the origin:

[6] *Treatise of plane geometry through geometric algebra*, p. 104.

$$\frac{6}{13}x+\frac{7}{13}y=0 \qquad \text{or} \qquad 6x+7y=0$$

This line intersects the third base line $-x-y+1=0$ at the point (7, –6), whose pencil of lines is described by:

$$a(-x-y+1)+(1-a)\left(\frac{6}{13}x+\frac{7}{13}y\right)=0$$

Then $2x+3y+4=0$ is the line of this pencil determined by $a=4/17$.

Three lines are concurrent if the determinant of their barycentric dual coordinates vanishes. On the other hand, the line at the infinity has dual coordinates [1/3, 1/3], that is, it is the centroid of the dual base.

Also the equation of any conic has a very simple form using barycentric coordinates:

$$\begin{pmatrix}1-x-y & x & y\end{pmatrix}\mathbf{S}\begin{pmatrix}1-x-y\\ x\\ y\end{pmatrix}=0$$

where **S** is a symmetric matrix. I have shown[7] that the matrix of the tangential conic (the conic plotted in the dual plane by the dual points corresponding to the tangents to the point conic) is equal to the inverse of the matrix of the point conic, its equation being:

$$\begin{bmatrix}1-a-b & a & b\end{bmatrix}\mathbf{S}^{-1}\begin{bmatrix}1-a-b\\ a\\ b\end{bmatrix}=0$$

where $[a, b]$ are Cartesian coordinates in the dual plane.

Scalar and exterior product in the plane

The *metric geometry* explained to our pupils is a skew geometry, that is, a censured geometry: only those problems involving scalar product are considered. However the scalar and exterior products arise in a very symmetric form in geometry. We may and must teach to our pupils both products:

$$v\cdot w=v_x w_x+v_y w_y \qquad v\wedge w=\left(v_x w_y-v_y w_x\right)e_{12}$$

Then the geometric product may be introduced with the help of complex numbers:

$$v\,w=v\cdot w+v\wedge w$$

[7] *Treatise of plane geometry through geometric algebra*, p. 133.

A light version may be to use a real, instead of imaginary, exterior product. Anyway it allows pupils to calculate areas of triangles without having their altitudes.

Plane trigonometry and the hyperbolic plane

Many years ago I had prepared this table where the identities of the circular and hyperbolic functions are compared. But then I could not imagine in which extent the geometric algebra develops the analogy between circular and hyperbolic trigonometry.

TRIGONOMETRIA 1-

CIRCULAR	HIPERBOLICA
$\sin^2 x + \cos^2 x = 1$	$\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1$
$\mathrm{tg}\, x = \frac{\sin x}{\cos x}$	$\mathrm{th}\, x = \frac{\mathrm{sh}\, x}{\mathrm{ch}\, x}$
$\sin(-x) = -\sin x$	$\mathrm{sh}(-x) = -\mathrm{sh}\, x$
$\cos(-x) = \cos x$	$\mathrm{ch}(-x) = \mathrm{ch}\, x$
$\mathrm{tg}(-x) = -\mathrm{tg}\, x$	$\mathrm{th}(-x) = -\mathrm{th}\, x$
SUMA D'ANGLES	
$\sin(x+y) = \sin x \cos y + \cos x \sin y$	$\mathrm{sh}(x+y) = \mathrm{sh}\, x\, \mathrm{ch}\, y + \mathrm{ch}\, x\, \mathrm{sh}\, y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$	$\mathrm{ch}(x+y) = \mathrm{ch}\, x\, \mathrm{ch}\, y + \mathrm{sh}\, x\, \mathrm{sh}\, y$
$\mathrm{tg}(x+y) = \frac{\mathrm{tg}\, x + \mathrm{tg}\, y}{1 - \mathrm{tg}\, x\, \mathrm{tg}\, y}$	$\mathrm{th}(x+y) = \frac{\mathrm{th}\, x + \mathrm{th}\, y}{1 + \mathrm{th}\, x\, \mathrm{th}\, y}$
ANGLES DOBLES	
$\sin 2x = 2 \sin x \cos x$	$\mathrm{sh}\, 2x = 2\, \mathrm{sh}\, x\, \mathrm{ch}\, x$
$\cos 2x = \cos^2 x - \sin^2 x$	$\mathrm{ch}\, 2x = \mathrm{ch}^2 x + \mathrm{sh}^2 x$
$\mathrm{tg}\, 2x = \frac{2\, \mathrm{tg}\, x}{1 - \mathrm{tg}^2 x}$	$\mathrm{th}\, 2x = \frac{2\, \mathrm{th}\, x}{1 + \mathrm{th}^2 x}$
ANGLES MEITAT	
$\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$	$\mathrm{sh}\, \frac{x}{2} = \pm\sqrt{\frac{\mathrm{ch}\, x - 1}{2}}$ + per $x > 0$, − per $x < 0$
$\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$	$\mathrm{ch}\, \frac{x}{2} = \sqrt{\frac{\mathrm{ch}\, x + 1}{2}}$
$\mathrm{tg}\, \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$	$\mathrm{th}\, \frac{x}{2} = \pm\sqrt{\frac{\mathrm{ch}\, x - 1}{\mathrm{ch}\, x + 1}} = \frac{\mathrm{ch}\, x - 1}{\mathrm{sh}\, x} = \frac{\mathrm{sh}\, x}{\mathrm{ch}\, x + 1}$
SUMA I RESTA DE FUNCIONS	
$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$	$\mathrm{sh}\, x + \mathrm{sh}\, y = 2\, \mathrm{sh}\, \frac{x+y}{2}\, \mathrm{ch}\, \frac{x-y}{2}$
$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$	$\mathrm{sh}\, x - \mathrm{sh}\, y = 2\, \mathrm{sh}\, \frac{x-y}{2}\, \mathrm{ch}\, \frac{x+y}{2}$
$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$	$\mathrm{ch}\, x + \mathrm{ch}\, y = 2\, \mathrm{ch}\, \frac{x+y}{2}\, \mathrm{ch}\, \frac{x-y}{2}$
$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$	$\mathrm{ch}\, x - \mathrm{ch}\, y = 2\, \mathrm{sh}\, \frac{x+y}{2}\, \mathrm{sh}\, \frac{x-y}{2}$
$\mathrm{tg}\, x \pm \mathrm{tg}\, y = \frac{\sin(x \pm y)}{\cos x \cos y}$	$\mathrm{th}\, x \pm \mathrm{th}\, y = \frac{\mathrm{sh}(x \pm y)}{\mathrm{ch}\, x\, \mathrm{ch}\, y}$

Figure 5

In the same way as complex numbers are those naturally associated with circular trigonometry, hyperbolic numbers are those naturally associated with hyperbolic trigonometry:

$$z = a + b\,e_{12} \quad e_{12}^2 = -1 \qquad\qquad z = a + b\,e_1 \quad e_1^2 = 1$$

So we have the analogous of Euler's and De Moivre's identities:

$$\exp(x\,e_{12}) = \cos x + e_{12}\sin x \qquad \exp(x\,e_1) = \cosh x + e_1 \sinh x$$

$$(\cos x + e_{12}\sin x)^n = \cos nx + e_{12}\sin nx \qquad (\cosh x + e_1\sinh x)^n = \cosh nx + e_1\sinh nx$$

A Euclidean angle is defined as quotient of the arc length divided by the radius of the circle (figure 6):

$$\alpha = \frac{s}{r} = \frac{2A}{r^2}$$

and it is proportional to the area of the circular sector.

In the same way a hyperbolic angle is defined as quotient of the arc length of the equilateral hyperbola $x^2 - y^2 = r^2$ divided by its radius r in the pseudo-Euclidean plane (figure 7):

$$\psi = \frac{s}{r} = \frac{2A}{r^2}$$

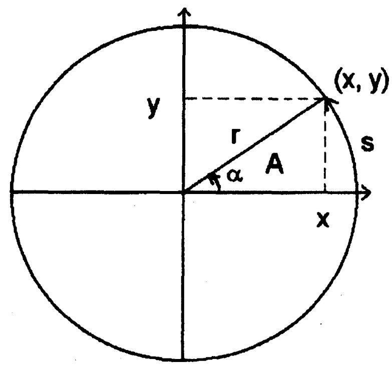


Figure 6

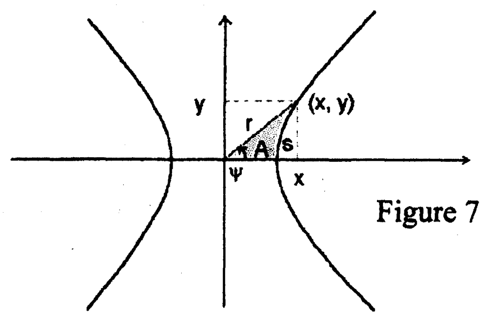


Figure 7

Of course, we cannot represent hyperbolic arguments with circle arcs as made in Euclidean triangles. So a new sketch of hyperbolic arguments by means of arcs of hyperbola is needed. In figure 8 we see a typical hyperbolic triangle suitably drawn to show that the addition of the three angles is $-\pi\,e_{12}$; and figure 9 shows the relation between arguments in different quadrants.

Figure 8

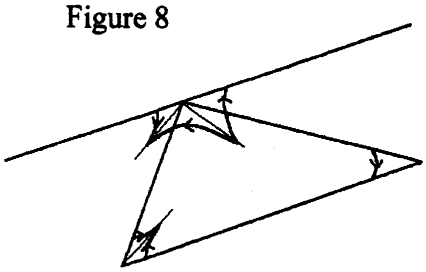
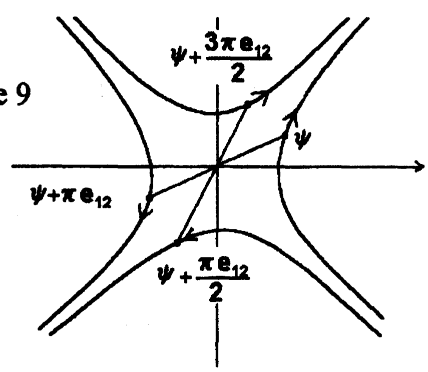


Figure 9



Also the laws of sines, cosines and tangents for a generic Euclidean triangle (figure 10):

$$\frac{|a|}{\sin\alpha}=\frac{|b|}{\sin\beta}=\frac{|c|}{\sin\gamma}$$

$$a^2=b^2+c^2-2|b||c|\cos\alpha$$

$$\frac{|a|+|b|}{|a|-|b|}=\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$$

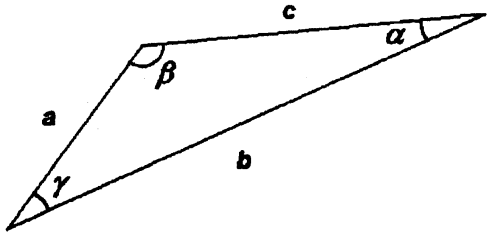


Figure 10

have their analogous laws for hyperbolic triangles:

$$\frac{|a|}{\sinh\alpha}=\frac{|b|}{\sinh\beta}=\frac{|c|}{\sinh\gamma}$$

$$a^2=b^2+c^2-2|b||c|\cosh\alpha$$

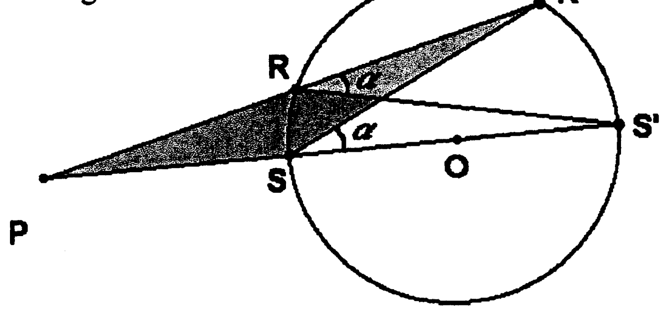
$$\frac{|a|+|b|}{|a|-|b|}=\frac{\tanh\frac{\alpha+\beta}{2}}{\tanh\frac{\alpha-\beta}{2}}$$

Some examples and tests showed me that they work with the suitable orientation of the hyperbolic arguments displayed in figure 9.

We also explain to our pupils the power of a point with respect to a circle, which is the value obtained when the coordinates of the point are substituted in the equation of the circle.

$$PR\ \ PR'=PS\ \ PS'=PO^2-OS^2=(x-x_0)^2+(y-y_0)^2-r^2$$

Figure 11



In the hyperbolic plane, the power of a point with respect to a hyperbola with constant radius is also constant. The proof is as follows: the yellow and blue hyperbolic triangles in figure 12 are similar because the hyperbolic argument ψ (in the hyperbolic plane) is constant and equal to a half of the arc length $R'S'$ of the hyperbola. Opposite similarity may be written using geometric product:

$$PR\,(PS')^{-1} = (PR')^{-1}\,PS$$

which implies:

$$PR'\,PR = PS\ \ PS' = PO^2 - OS^2 = (x - x_0)^2 - (y - y_0)^2 - r^2$$

Observe that the power of a point with respect to a hyperbola in the hyperbolic plane is just obtained using the Cartesian equation! This clearly shows that Cartesian coordinates do not necessarily mean Euclidean coordinates. Leibniz already criticized the incompleteness of Cartesian coordinates[8].

I also took figure 12 for the cover of my *Treatise of plane geometry through geometric algebra.*

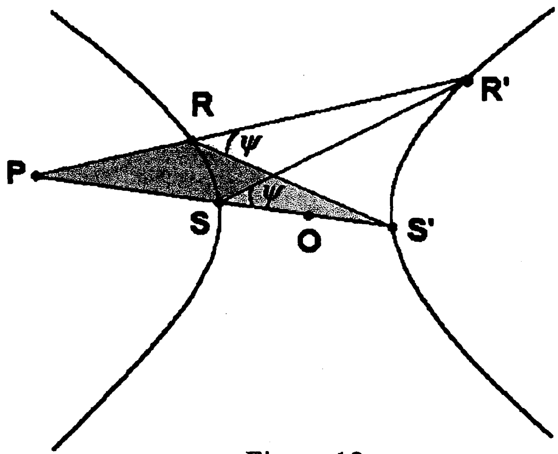


Figure 12

Rank of a matrix, exterior product and systems of equations

We explain to our pupils that the rank of a matrix is the number of linearly independent rows or columns. In the method that uses determinants to find the rank, we take an element of the matrix and we add rows and columns always taking that matrix having non-null determinant:

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 1 & 3 & 2 \\ -3 & 1 & 0 & 5 \end{pmatrix}$$

[8] "Je puis demonstrer que ce que j'ay avancé suit de ma caracteristique lineare ou geometrique dont je vous envoyé un essay. Car premierement je puis exprimer parfaitement par ce calcul toute la nature ou definition de la figure (ce que l'Algebre ne fait jamais, car disant que $x^2 + y^2$ aeq. a^2 est l'equation du cercle, il faut expliquer par la figure ce que c'est que ce x et y, c'est à dire que ce sont des lignes droites, dont l'une est perpendiculaire à l'autre et l'une commence par le centre, l'autre par la circonference de la figure)." (letter to Christian Huygens without date but likely written by Leibniz in December 1680) *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern* vol.VII, p. 580. Herausgegeben von C. I. Gerhardt, George Olms Verlagsbuchhandlung (Hildesheim, 1962).

$$1 \neq 0 \rightarrow \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -2 \neq 0 \rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ -3 & 1 & 0 \end{vmatrix} = 0 \rightarrow \begin{vmatrix} 1 & 1 & 3 \\ 3 & 1 & 2 \\ -3 & 1 & 5 \end{vmatrix} = 0 \quad \Rightarrow \quad \operatorname{rank} \mathbf{M} = 2$$

If all the determinants obtained adding one row and one column to a non-null n-dimensional determinant are zero then all the other n+1 dimensional determinants are null. This ensures that the rank of the matrix is equal to n.

In the alternative method using exterior product we have:

$$(e_1 + e_2 + 2e_3 + 3e_4) \wedge (3e_1 + e_2 + 3e_3 + 2e_4) =$$

$$= -2e_1 \wedge e_2 - 3e_1 \wedge e_3 - 7e_1 \wedge e_4 + e_2 \wedge e_3 - e_2 \wedge e_4 - 5e_3 \wedge e_4$$

The coefficients of the exterior product are the corresponding minors of the matrix. Now we take the exterior product with the third vector:

$$(e_1 + e_2 + 2e_3 + 3e_4) \wedge (3e_1 + e_2 + 3e_3 + 2e_4) \wedge (-3e_1 + e_2 + 5e_4) = 0$$

So we conclude that the rank of the matrix **M** is 2.

Note that the determinants of a given order n, which are the components of the exterior product of n vectors are not linearly independent although they are orthogonal from the point of view of the Pythagorean theorem. I solely wish to manifest that the following words are not synonyms in geometric algebra: *linearly independent* $\neq$ *orthogonal* $\neq$ *perpendicular*. Further explanations can be provided if required.

Another application of the exterior product is the resolution of systems of equations.

$$\begin{cases} a_{11}x_1 + & \cdots & + a_{1n}x_n = b_1 \\ \vdots & & \vdots \\ a_{n1}x_1 + & \cdots & + a_{nn}x_n = b_n \end{cases}$$

which may be written as a vectorial equality:

$$x_1 v_1 + \cdots + x_n v_n = b$$

where $v_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. In order to solve it, we take the exterior product with all the other vectors:

$$x_i \; v_1 \wedge v_2 \wedge \cdots \wedge v_n = v_1 \wedge \cdots \wedge v_{i-1} \wedge b \wedge v_{i+1} \wedge \cdots \wedge v_n$$

whence it follows Cramer's rule:

$$x_i = \frac{v_1 \wedge \cdots \wedge v_{i-1} \wedge b \wedge v_{i+1} \wedge \cdots \wedge v_n}{v_1 \wedge v_2 \wedge \cdots \wedge v_n} = \frac{\det(v_1 \cdots v_{i-1}\ b\ v_{i+1} \cdots v_n)}{\det(v_1\ v_2 \cdots v_n)}$$

Change of coordinates

One set of coordinates may be easily changed to another set of coordinates with the exterior product. For example, let us see the change from Cartesian to spherical coordinates:

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$dx = \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi$$

$$dy = \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi$$

$$dz = \cos\theta\, dr - r\sin\theta\, d\theta$$

When taking the exterior product of the three differentials we obtain the volume element:

$$dV = dx \wedge dy \wedge dz = r^2 \sin\theta\, dr \wedge d\theta \wedge d\varphi$$

But we can apply it to any geometric element such as the surface element:

$$dx \wedge dy = r\sin^2\theta\, dr \wedge d\varphi + r^2 \sin\theta\cos\theta\, d\theta \wedge d\varphi$$

$$dy \wedge dz = -r\sin\varphi\, dr \wedge d\theta + r^2\sin^2\theta\cos\varphi\, d\theta \wedge d\varphi + r\sin\theta\cos\theta\cos\varphi\, d\varphi \wedge dr$$

$$dz \wedge dx = r\cos\varphi\, dr \wedge d\theta + r^2\sin^2\theta\sin\varphi\, d\theta \wedge d\varphi + r\sin\theta\cos\theta\sin\varphi\, d\varphi \wedge dr$$

$$dA = dx \wedge dy + dy \wedge dz + dz \wedge dx$$

This is a vectorial equation whence one deduces the modulus of the element of area:

$$|dA|^2 = |dx \wedge dy|^2 + |dy \wedge dz|^2 + |dz \wedge dx|^2 =$$

$$= r^2 |dr \wedge d\theta|^2 + r^4 \sin^2\theta\, |d\theta \wedge d\varphi|^2 + r^2 \sin^2\theta\, |d\varphi \wedge dr|^2$$

So we have the differential of area in spherical coordinates:

$$dA = r\, dr \wedge d\theta + r^2 \sin\theta\, d\theta \wedge d\varphi + r\sin\theta\, d\varphi \wedge dr$$

Rotations and symmetries

The axial symmetry (or reflection) shown in figure 13 may be expressed in the form:

$$PR' = v^{-1} PR\, v$$

where v is the direction vector of the axis of symmetry. This is easily proved because the perpendicular component anticommutes with the vector v:

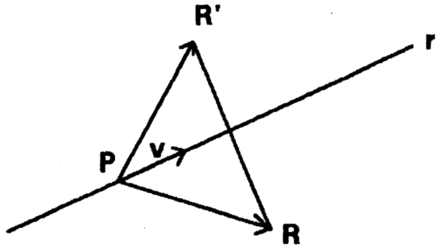
$$v^{-1}\left(PR_{\perp} + PR_{\parallel}\right)v = v^{-1}v\left(-PR_{\perp} + PR_{\parallel}\right) = -PR_{\perp} + PR_{\parallel}$$

If the point R belongs to the line, the vector PR and the direction vector v are proportional and commute:

$$PR\, v = v\, PR \quad \Leftrightarrow \quad PR\, v - v\, PR = 0 \quad \Leftrightarrow \quad PR \wedge v = 0 \quad \Leftrightarrow \quad PR = v^{-1}\, PR\, v$$

The last equation is the *algebraic equation* of a line and shows that vector PR remains invariant under a reflection in the direction of the line. That is, the point R only belongs to the line when it coincides with the point reflected in this line. Separating components we have:

Figure 13



$$\frac{x - x_P}{v_1} = \frac{y - y_P}{v_2}$$

Also, a rotation of angle α may be expressed in a way analogous to axial symmetries:

$$v' = v\left(\cos\alpha + e_{12}\sin\alpha\right) = \left(\cos\alpha/2 - e_{12}\sin\alpha/2\right)v\left(\cos\alpha/2 + e_{12}\sin\alpha/2\right)$$

The first form is only valid on a plane, while the second form is general for any dimension. If we take any complex number with argument $\alpha/2$ we may write:

$$v' = z^{-1} v\, z \qquad\qquad z = |z|_{\alpha/2}$$

Now we may easily prove which transformation is the composition of two reflections in the directions v and w:

$$t' = v^{-1} t\, v$$

$$t'' = w^{-1} t'\, w = w^{-1} v^{-1} t\, v\, w = z^{-1} v\, z$$

$$z = v\, w$$

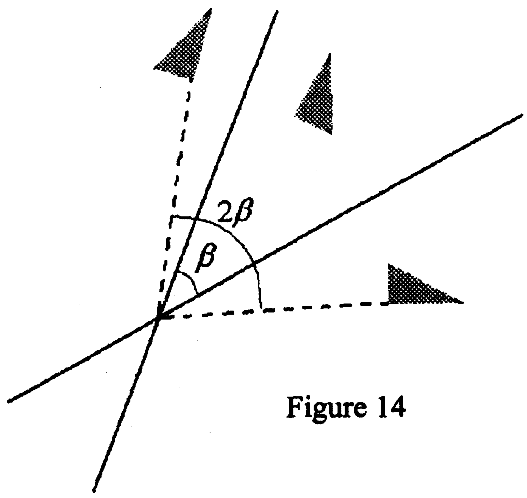


Figure 14

The product of two vectors is a complex number and therefore the composition of two reflections is a rotation over double the angle between both directions (figure 14). If the directions are parallel, the rotation has center at the infinity and becomes a translation along a distance double of that between the axes of symmetry.

Of course, the composition of two rotations on the plane is another rotation over an angle equal to the addition of both angles as follows from the product of complex numbers in polar form:

$$z\,t = |z|_\alpha\,|t|_\beta = |z|\,|t|_{\alpha+\beta}$$

In the Euclidean space, each rotation is described in the following way:

$$v' = (\cos\alpha/2 - u\sin\alpha/2)\,v\,(\cos\alpha/2 + u\sin\alpha/2)$$

where u is a unitary bivector. In general:

$$v' = q^{-1} v\, q$$

where q is any quaternion whose plane is the plane of rotation and whose argument is half of the rotation angle. This expression is applied to any element of the algebra of three-dimensional space such as scalars, vectors, bivectors or volumes. However in technological applications the use of only bivectors and quaternions is preferred.

The composition of two rotations in any planes is obtained through the product of quaternions:

$$v'' = r^{-1} q^{-1} v\, q\, r$$

This is the best way to describe rotations and composition of rotations with immediate technological interest, such as engineers have discovered long ago.

However there is a difficulty with mirror reflections, which cannot be written in this way. This has a physical consequence: an asymmetric molecule cannot be converted into its mirror image, as Louis Pasteur showed in the case of tartaric acid[9].

Notable points of a triangle

The conditions of intersection of the medians, bisectors of the sides, the angle bisectors and the altitudes lead to geometric equations for the notable points of a triangle PQR whose solutions are respectively[10]:

$$G = \frac{P+Q+R}{3} \qquad \text{(centroid)}$$

$$O = -(P^2QR + Q^2RP + R^2PQ)(2\,PQ \wedge QR)^{-1} \qquad \text{(circumcenter)}$$

$$I = \frac{P\,|QR| + Q\,|RP| + R\,|PQ|}{|QR| + |RP| + |PQ|} \qquad \text{(incenter)}$$

[9] L. Pasteur, *Researches on the Molecular Asymmetry of Natural Organic Products* (1860), p. 24. Published by the Alambic Club (Edinburgh, 1905).

[10] *Treatise of plane geometry through geometric algebra*, chap. 8.

$$H = (P\ P \cdot QR + Q\ Q \cdot RP + R\ R \cdot PQ)(QR \wedge RP)^{-1}$$ (orthocenter)

$$N = (P\,QR\,P + Q\,RP\,Q + R\,PQ\,R)(4\,PQ \wedge QR)^{-1}$$ (center of the nine-point circle)

Observe that the points lying on the Euler line (O, H, N) have expressions implying the geometric (Clifford) product (it is not needed for the centroid G but we may also add a factor and a divisor $(PQ \wedge QR)^{-1}$), expressions which cannot be written using only the scalar product of the so-called *metric geometry*. This clearly shows the importance of the geometric product in geometry and the partial vision and censure with which geometry is nowadays taught.

Some comments on the Lobachevskian geometry

Also Lobachevsky's geometry may be easily explained to our pupils using geometric algebra. The two-sheeted hyperboloid with constant radius in the pseudo-Euclidean space realizes Lobachevsky's geometry:

$$z^2 - x^2 - y^2 = 1$$

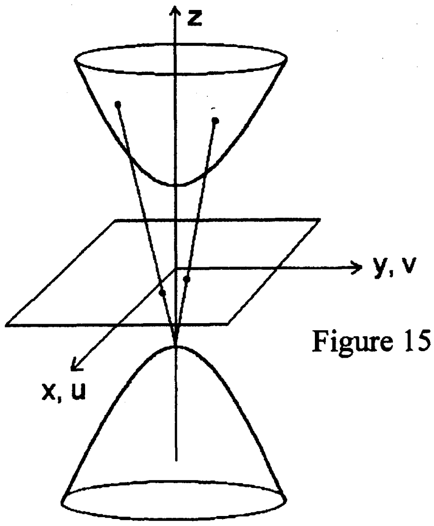
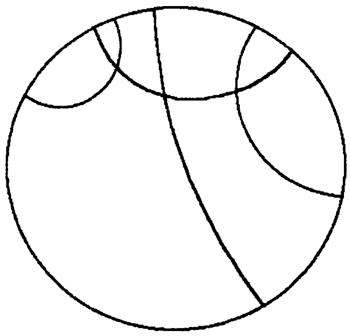


Figure 15

Starting from this point we may deduce expressions of the change of coordinates for any projection.

Let us see, for instance, the stereographic projection or Poincaré's disk with more detail. This conformal projection whose point of view is the pole of one sheet is displayed in figures 15 and 16:

Figure 16



$$\frac{x}{u} = z + 1 \qquad \frac{y}{v} = z + 1$$

where u, v are the coordinates on the plane of projection. The arc length is:

$$ds = \sqrt{dx^2 + dy^2 - dz^2} = \frac{2\sqrt{du^2 + dv^2}}{1 - u^2 - v^2}$$

The exterior product allows us to calculate quickly the differential of area:

$$dA = \sqrt{(dx \wedge dy)^2 - (dy \wedge dz)^2 - (dz \wedge dx)^2} = \frac{4\,du \wedge dv}{(1 - u^2 - v^2)^2}$$

Also with the help of exterior product we may deduce the azimuthal equivalent projection, which preserves areas, by imposing that the modulus of the area differentials with Cartesian coordinates and in the projection must be equal:

$$\begin{cases} dA^2 = \dfrac{1}{z^2}(dx \wedge dy)^2 = (du \wedge dv)^2 \\ u = x\, f(z) \qquad v = y\, f(z) \end{cases}$$

$$u = x\sqrt{\frac{2}{z+1}} \qquad v = y\sqrt{\frac{2}{z+1}}$$

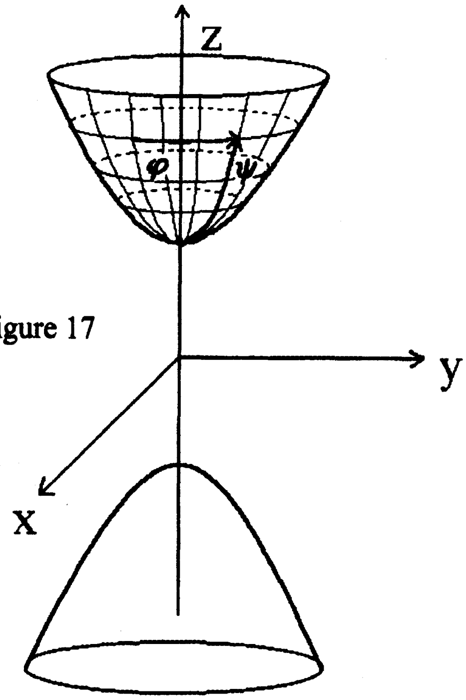


Figure 17

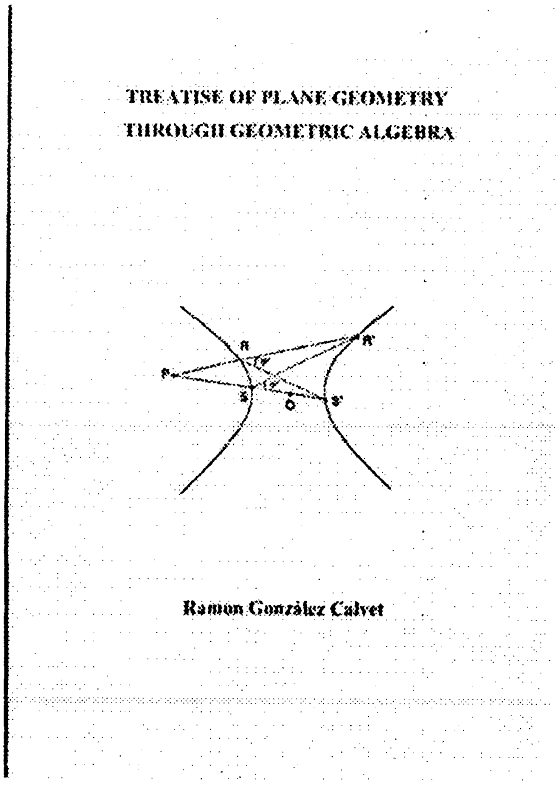
Other projections of the hyperboloid are the central projection (Beltrami's disk), whose point of view is the origin of coordinates, the cylindrical equidistant projection, which uses Weierstrass' coordinates (figure 17) analogous of spherical coordinates:

$$x = \sinh\psi \cos\varphi \qquad y = \sinh\psi \sin\varphi$$

$$z = \cosh\psi$$

$$ds^2 = d\psi^2 + \sinh^2\psi\, d\varphi^2$$

$$dA = \sinh\psi\, d\psi \wedge d\varphi$$



the cylindrical conformal projection (analogous of Mercator's projection) and the conic equidistant and conformal projections[11]. For all of them the exterior product allows us to calculate the area differential.

The *Treatise of plane geometry through geometric algebra*

As a pedagogic tool for the introduction in geometric algebra, you have available the *Treatise of plane geometry through geometric algebra.* This book has four parts:

1. The vector plane and complex numbers
2. Geometry of the Euclidean plane.
3. Pseudo-Euclidean geometry.

[11] *Treatise of plane geometry through geometric algebra*, p. 203.

4. Plane projections of three-dimensional spaces.

and many solved exercises in each lesson. It may be used as a reference book for high school pupils, but also as a textbook for introductory courses on geometric algebra or preliminary courses on plane geometry at the first university year. In fact, my aim when writing it was to be a bridge between high school and university and help for teachers. In this book are collected a significant part of the lessons given in the summer courses on geometric algebra for teachers that we (Josep Manel Parra and me) have imparted in the framework of the *Escola d'Estiu de Secundària* of the Col·legi de Doctors i Llicenciats en Filosofia i Lletres i Ciències de Catalunya[12]. Perhaps you believe that I am selling the book but in any case I do not do business since you may freely download it from the Internet site:

http://campus.uab.es/~PC00018

Thank you very much for listening to me and I will answer your questions as best as I can.

[12] Its homepage is: http://www.cdlcat.es